Lecture 4
Integral Transport Equations

1 A 1-D Integral Equation for the Angular Flux

The primary purpose of this section is to derive an integral equation for the angular flux in 1-D slab geometry under the assumptions of isotropic scattering, an isotropic distributed source, isotropic incident angular fluxes, and constant cross-sections. These assumptions are not necessary, but they result in considerable simplifications. We begin by solving the following equation on the interval $[x_L, x_R]$:

$$\mu \frac{\partial \psi}{\partial x} + \sigma_t \psi = \frac{\sigma_s}{4\pi}\, \phi(x) + \frac{Q_0}{4\pi}\,, \tag{1}$$

under the further assumption that $\psi(x_L, \mu) = \frac{\phi_L}{4\pi}$ for $\mu > 0$, and that $\psi(x_R, \mu) = \frac{\phi_R}{4\pi}$ for $\mu < 0$. Using the integrating factor approach we proceed as follows:

$$\begin{aligned}
\frac{\partial \psi}{\partial x} + \frac{\sigma_t}{\mu}\, \psi &= \frac{\sigma_s \phi + Q_0}{4\pi\mu}\,, \\
\exp\left(\frac{\sigma_t x}{\mu}\right) \frac{\partial \psi}{\partial x} + \exp\left(\frac{\sigma_t x}{\mu}\right) \frac{\sigma_t x}{\mu}\, \psi &= \exp\left(\frac{\sigma_t x}{\mu}\right) \frac{\sigma_s \phi + Q_0}{4\pi\mu}\,, \\
\frac{\partial}{\partial x}\left[\exp\left(\frac{\sigma_t x}{\mu}\right) \psi\right] &= \exp\left(\frac{\sigma_t x}{\mu}\right) \frac{\sigma_s \phi + Q_0}{4\pi\mu}\,.
\end{aligned} \tag{2}$$

Integrating Eq. (2) from x_L to x for $\mu > 0$, we obtain

$$\psi(x,\mu)\exp\left(\frac{\sigma_t x}{\mu}\right) - \frac{\phi_L}{4\pi}\exp\left(\frac{\sigma_t x_L}{\mu}\right) = \int_{x_L}^{x} \exp\left(\frac{\sigma_t x'}{\mu}\right) \frac{\sigma_s\phi(x') + Q_0(x')}{4\pi\mu}\, dx'\,.$$

Integrating Eq. (2) from x to x_R for $\mu < 0$, we obtain

$$\frac{\phi_R}{4\pi}\exp\left(\frac{\sigma_t x_R}{\mu}\right) - \psi(x,\mu)\exp\left(\frac{\sigma_t x}{\mu}\right) = \int_{x}^{x_R} \exp\left(\frac{\sigma_t x'}{\mu}\right) \frac{\sigma_s\phi(x') + Q_0(x')}{4\pi\mu}\, dx'\,.$$

So the final solution for $\mu > 0$ is

$$\psi(x,\mu) = \frac{\phi_L}{4\pi}\exp\left[\frac{\sigma_t}{\mu}(x_L - x)\right] + \int_{x_L}^{x} \exp\left[\frac{\sigma_t}{\mu}(x' - x)\right] \frac{\sigma_s\phi(x') + Q_0(x')}{4\pi\mu}\, dx'\,, \tag{3}$$

and the final solution for $\mu < 0$ is

$$\psi(x,\mu) = \frac{\phi_R}{4\pi}\exp\left[\frac{\sigma_t}{\mu}(x_R - x)\right] - \int_{x}^{x_R} \exp\left[\frac{\sigma_t}{\mu}(x' - x)\right] \frac{\sigma_s\phi(x') + Q_0(x')}{4\pi\mu}\, dx'\,. \tag{4}$$

Note that that Eqs. (3) and (4) actually constitute an integral equation for the angular flux since the scalar flux is a function of the angular flux. In principal, one can iteratively solve this equation using the "order-of-scatter" approach described in Lecture 3.

2 A 1-D Integral Equation for the Scalar Flux

The purpose of this section is to derive a 1-D integral equation for the scalar flux. We start with the 1-D integral equation for the angular flux. In particular, we first integrate Eq. (3)

over all $\mu > 0$:

$$\begin{aligned}
\phi^+(x) &= 2\pi \int_0^1 \frac{\phi_L}{4\pi} \exp\left[\frac{\sigma_t}{\mu}(x_L - x)\right] d\mu + \\
&\quad 2\pi \int_0^1 \int_{x_L}^{x} \exp\left[\frac{\sigma_t}{\mu}(x' - x)\right] \frac{\sigma_s \phi(x') + Q_0(x')}{4\pi\mu}\, dx'\, d\mu\,, \qquad (5)
\end{aligned}$$

where ϕ^+ denotes the contribution to the scalar flux from $\mu > 0$. To evaluate the angular integrals in Eq. (5), we make the substitution $z = \frac{1}{\mu}$. Then $d\mu = -z^{-2}dz$, so

$$\begin{aligned}
\phi^+(x) &= \int_1^\infty \frac{\phi_L}{2} \exp\left[\sigma_t(x_L - x)z\right] z^{-2}\, dz + \\
&\quad \int_1^\infty \left[\int_{x_L}^{x} \exp\left[\sigma_t(x' - x)z\right] \frac{\sigma_s \phi(x') + Q_0(x')}{2z}\, dx'\right] dz \\
&= \int_1^\infty \frac{\phi_L}{2} \exp\left[\sigma_t(x_L - x)z\right] z^{-2}\, dz + \\
&\quad \int_{x_L}^{x} \frac{\sigma_s \phi(x') + Q_0(x')}{2} \left[\int_1^\infty \exp\left[-\sigma_t(x' - x)z\right] z^{-1}\, dz\right] dx' \\
&= \phi_L \frac{1}{2} E_2\left[\sigma_t(x - x_L)\right] + \int_{x_L}^{x} \left(\sigma_s \phi(x') + Q_0(x')\right) \frac{1}{2} E_1\left[\sigma_t(x' - x)\right]\, dx'\,, \qquad (6)
\end{aligned}$$

where

$$E_n(x) = \int_1^\infty \exp\left(-xz\right) z^{-n}\, dz \quad \text{, for all non-negative } n. \qquad (7)$$

The family of functions, $E_n(x)$, are called the exponential integrals. Their properties are described in Appendix A.

We next integrate Eq. (4) over all $\mu < 0$:

$$\begin{aligned}
\phi^-(x) &= 2\pi \int_{-1}^0 \frac{\phi_R}{4\pi} \exp\left[\frac{\sigma_t}{\mu}(x_R - x)\right] d\mu - \\
&\quad 2\pi \int_{-1}^0 \int_{x}^{x_R} \exp\left[\frac{\sigma_t}{\mu}(x' - x)\right] \frac{\sigma_s \phi(x') + Q_0(x')}{4\pi\mu}\, dx'\, d\mu\,, \qquad (8)
\end{aligned}$$

where ϕ^- denotes the contribution to the scalar flux from $\mu < 0$. To evaluate the angular integrals in Eq. (5), we make the substitution $z = -\frac{1}{\mu}$. Then $d\mu = z^{-2}dz$, and we eventually obtain

$$\phi^-(x) = \phi_R \frac{1}{2} E_2 \left[\sigma_t(x_R - x)\right] + \int_x^{x_R} \left(\sigma_s \phi(x') + Q_0(x')\right) \frac{1}{2} E_1 \left[\sigma_t(x' - x)\right] \, dx' \,, \tag{9}$$

Adding Eqs. (6) and (9), we obtain the desired integral equation:

$$\begin{aligned} \phi(x) - \int_{x_L}^{x_R} \sigma_s \phi(x') \frac{1}{2} E_1 \left(\sigma_t |x' - x|\right) \, dx' &= \phi_L \frac{1}{2} E_2 \left[\sigma_t(x - x_L)\right] + \\ \phi_R \frac{1}{2} E_2 \left[\sigma_t(x_R - x)\right] &+ \int_{x_L}^{x_R} Q_0(x') \frac{1}{2} E_1 \left(\sigma_t |x' - x|\right) \, dx' \,, \end{aligned} \tag{10}$$

The assumption of isotropic scattering is necessary to obtain an integral equation for the scalar flux that contains only the scalar flux itself.

3 A 3-D Integral Equation for the Angular Flux

The purpose of this section is to derive a 3-D integral equation for the angular flux. We begin with the 3-D integro-differential transport equation. We will initially assume constant cross-sections but admit the possibility of an anisotropic total source (scattering plus inhomogeneous):

$$\overrightarrow{\Omega} \cdot \overrightarrow{\nabla} \psi(\overrightarrow{r}, \overrightarrow{\Omega}) + \sigma_t \psi(\overrightarrow{r}, \overrightarrow{\Omega}) = \mathcal{Q}(\overrightarrow{r}, \overrightarrow{\Omega}) \,, \tag{11}$$

where $\mathcal{Q}$ denotes the total source. From basic calculus, we know that the operator, $\overrightarrow{\Omega} \cdot \overrightarrow{\nabla}$ represents the directional derivative in the direction $\overrightarrow{\Omega}$. We will now exploit this fact by defining a local coordinate system about the point $\overrightarrow{r}$, as shown in Fig. 1. The local coordinates are defined at the point, $\overrightarrow{r}$, and consist of a pathlength variable, s, and the direction, $\overrightarrow{\Omega}$. In particular,

$$\psi(s, \overrightarrow{\Omega}) \equiv \psi(\overrightarrow{r}_s, \overrightarrow{\Omega}), \tag{12}$$

where

$$\overrightarrow{r}_s = \overrightarrow{r} - s\overrightarrow{\Omega}. \tag{13}$$

There are several things to note here. The first is that the angular variable, $\overrightarrow{\Omega}$, is playing a dual role as a spatial and angular variable. The second is that we have defined s so that it increases along the direction $-\overrightarrow{\Omega}$ rather than $\overrightarrow{\Omega}$. This follows from the fact that we want positive values of s to correspond to points, $\overrightarrow{r}_s$, that are *upstream* of the point, $\overrightarrow{r}$, because only the upstream points contribute to the angular flux solution at $\overrightarrow{r}$ in the direction, $\overrightarrow{\Omega}$. This orientation of the coordinate, s, is not really necessary, but it makes the derivation easier to understand. Transforming Eq. (10) to the local frame, we obtain

$$-\frac{\partial}{\partial s}\psi(s, \overrightarrow{\Omega}) + \sigma_t\psi(s, \overrightarrow{\Omega}) = \mathcal{Q}(s, \overrightarrow{\Omega}). \tag{14}$$

Multiplying Eq. (14) by -1, we get

$$\frac{\partial}{\partial s}\psi(s, \overrightarrow{\Omega}) - \sigma_t\psi(s, \overrightarrow{\Omega}) = -\mathcal{Q}(s, \overrightarrow{\Omega}).$$

Multiplying the above equation by the integrating factor, $\exp(-\sigma_t s)$, we obtain

$$\frac{\partial}{\partial s}\left\{\psi(s, \overrightarrow{\Omega})\exp(-\sigma_t s)\right\} = -\mathcal{Q}(s, \overrightarrow{\Omega})\exp(-\sigma_t s),$$

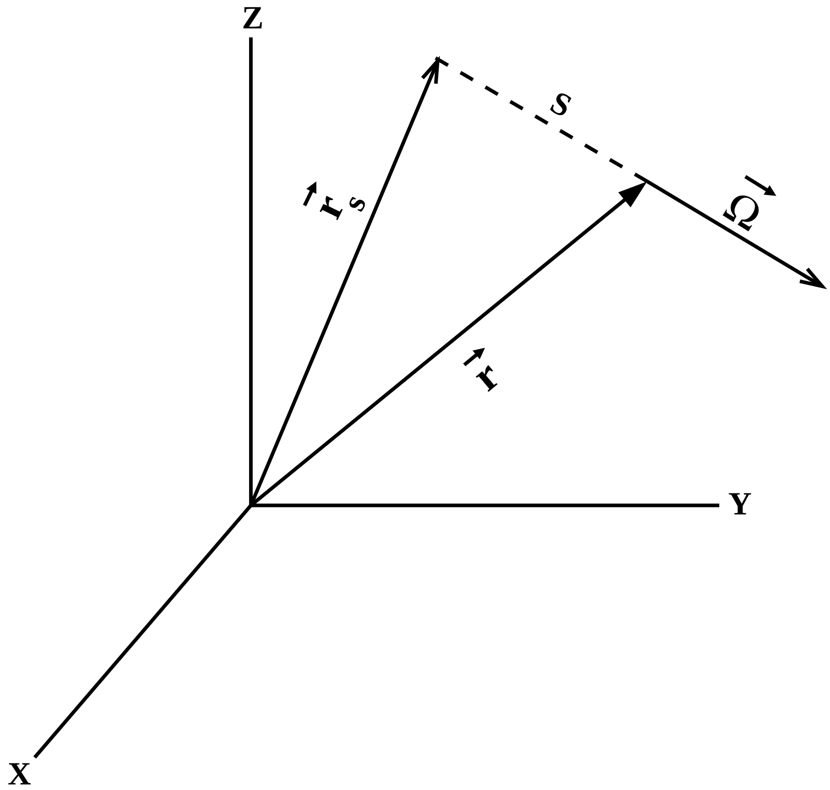

Figure 1: The local spatial coordinate system at the point $\overrightarrow{r}$. Note that $\overrightarrow{r}_s = \overrightarrow{r} - s\overrightarrow{\Omega}$. This is a form of local spherical coordinate system with $\overrightarrow{\Omega}$ playing the role of both a spatial and angular variable.

The function, $s_b(\overrightarrow{r}, \overrightarrow{\Omega})$, is defined to be the distance from point $\overrightarrow{r}$ to the outer boundary of the domain along the direction $-\overrightarrow{\Omega}$. Integrating the previous equation in s from $s = 0$ to $s = s_b$, we get

$$\psi(s_b, \overrightarrow{\Omega}) \exp(-\sigma_t s_b) - \psi(0, \overrightarrow{\Omega}) = -\int_0^{s_b} \mathcal{Q}(s, \overrightarrow{\Omega}) \exp(-\sigma_t s)\, ds\,.$$

Remembering that $s = 0$ corresponds to the point $\overrightarrow{r}$, we solve the previous equation for ψ at that point:

$$\psi(\overrightarrow{r}, \overrightarrow{\Omega}) = \psi(s_b, \overrightarrow{\Omega}) \exp(-\sigma_t s_b) + \int_0^{s_b} \mathcal{Q}(s, \overrightarrow{\Omega}) \exp(-\sigma_t s) \, ds \, .$$

Adding the explicit global spatial dependence in the previous equations yields our desired integral equation for the angular flux:

$$\psi(\overrightarrow{r}, \overrightarrow{\Omega}) = \psi(\overrightarrow{r} - s_b \overrightarrow{\Omega}, \overrightarrow{\Omega}) \exp(-\sigma_t s_b) + \int_0^{s_b} \mathcal{Q}(\overrightarrow{r} - s_b \overrightarrow{\Omega}, \overrightarrow{\Omega}) \exp(-\sigma_t s) \, ds \, . \quad (15)$$

This is an integral equation because the total source includes the scattering source, which is a function of the angular flux itself. We can account for spatially-dependent cross-sections by noting that the integrating factor for this case is

$$\exp\left[-\int_0^s \sigma_t(s') ds'\right] ,$$

and that Eq. (15) becomes

$$\begin{aligned} \psi(\overrightarrow{r}, \overrightarrow{\Omega}) \quad = \quad & \psi(\overrightarrow{r} - s_b \overrightarrow{\Omega}, \overrightarrow{\Omega}) \exp\left[-\int_0^{s_b} \sigma_t(s') \, ds'\right] + \\ & \int_0^{s_b} \mathcal{Q}(\overrightarrow{r} - s \overrightarrow{\Omega}, \overrightarrow{\Omega}) \exp\left[-\int_0^{s} \sigma_t(s') \, ds'\right] \, ds \, . \end{aligned} \quad (16)$$

Equation (16) reveals the fundamental nature of radiation transport. The angular flux solution at a point, $\overrightarrow{r}$, in the direction, $\overrightarrow{\Omega}$, arises entirely from the incident flux in direction $\overrightarrow{\Omega}$ and the sources in direction $\overrightarrow{\Omega}$ at points *upstream* of $\overrightarrow{r}$ on the *line* corresponding to

$[0, s]$. This line is called the *characteristic.* The incident flux makes a contribution to the solution that is attenuated in proportion to the total (integrated) number of mean-free-paths between the incident flux and the solution point. Similarly, the source at each point on the characteristic makes a differential contribution to the solution that is attenuated in proportion to the total (integrated) number of mean-free-paths between the source and the solution point. Note that in a sourceless void, the angular flux solution is simply equal to the incident angular flux. Furthermore, it is not difficult to see that, in a sourceless void, the angular flux solution is *constant* along each and every *characteristic.*

4 A 3-D Integral Equation for the Scalar Flux

To obtain an integral equation for the scalar flux, we need simply assume isotropic scattering and integrate Eq. (16) over all directions. For simplicity, we assume an isotropic total source:

$$\begin{aligned} \phi(\overrightarrow{r}) &= \int_{4\pi} \psi(\overrightarrow{r} - s_b \overrightarrow{\Omega}, \overrightarrow{\Omega}) \exp\left[- \int_0^{s_b} \sigma_t(s')\, ds' \right] d\Omega + \\ & \int_{4\pi} \int_0^{s_b} \frac{Q_0(\overrightarrow{r} - s\overrightarrow{\Omega})}{4\pi} \exp\left[- \int_0^{s} \sigma_t(s')\, ds' \right] ds\, d\Omega\,. \end{aligned} \tag{17}$$

However, we can obtain a much more interesting form of this equation if we assume zero incident fluxes and spatially-constant cross-sections:

$$\phi(\overrightarrow{r}) = \int_{4\pi} \int_0^{s_b} \frac{Q_0(\overrightarrow{r} - s\overrightarrow{\Omega})}{4\pi} \exp\left(-\sigma_t s\right) \, ds\, d\Omega\,.$$

We first divide and multiply the above equation by s^2:

$$\phi(\overrightarrow{r}) = \int_{4\pi} \int_0^{s_b} \frac{\mathcal{Q}_0(\overrightarrow{r} - s\overrightarrow{\Omega})}{4\pi s^2} \exp(-\sigma_t s)\, s^2 \; ds\, d\Omega\,. \tag{18}$$

Note that our local spatial coordinate system is actually a spherical coordinate system, and that the differential volume associated with point $\overrightarrow{r}_s$ is

$$dV = s^2\, ds\, d\Omega\,.$$

Because of the dual role played by $\overrightarrow{\Omega}$, the integration over direction is also an integration over space. Thus we can re-express Eq. (18) as follows:

$$\phi(\overrightarrow{r}) = \int_{\mathcal{D}} \frac{\mathcal{Q}_0(\overrightarrow{r}')}{4\pi\|\overrightarrow{r}' - \overrightarrow{r}\|^2} \exp\left(-\sigma_t\|\overrightarrow{r}' - \overrightarrow{r}\|\right)\, dV'\,, \tag{19}$$

where $\mathcal{D}$ denotes the problem domain. This is the well-known “point-kernel” equation for volumetric isotropic sources.

Another useful kernel is the point kernel for volumetric anisotropic sources. The derivation is identical to that for the isotropic point kernel except that one begins with an anisotropic source rather than an isotropic source, i.e.,

$$\phi(\overrightarrow{r}) = \int_{4\pi} \int_0^{s_b} \mathcal{Q}(\overrightarrow{r} - s\overrightarrow{\Omega}, \overrightarrow{\Omega}) \exp(-\sigma_t s)\; ds\, d\Omega\,.$$

The final result is

$$\phi(\overrightarrow{r}) = \int_{\mathcal{D}} \frac{\mathcal{Q}(\overrightarrow{r}', \overrightarrow{\Omega}_0)}{\|\overrightarrow{r}' - \overrightarrow{r}\|^2} \exp\left(-\sigma_t\|\overrightarrow{r}' - \overrightarrow{r}\|\right)\, dV'\,, \tag{20a}$$

where

$$\overrightarrow{\Omega}_0 = \frac{\overrightarrow{r} - \overrightarrow{r}'}{\| \overrightarrow{r} - \overrightarrow{r}' \|} \,. \tag{20b}$$

With a space-dependent cross-section, Eq. (20a) becomes

$$\phi(\overrightarrow{r}) = \int_{\mathcal{D}} \frac{\mathcal{Q}(\overrightarrow{r}', \overrightarrow{\Omega}_0)}{\| \overrightarrow{r}' - \overrightarrow{r} \|^2} \exp\left[-\tau(\overrightarrow{r}', \overrightarrow{r}) \right] \, dV' \,, \tag{21a}$$

where $\tau(\overrightarrow{r}', \overrightarrow{r})$ represents the total number of mean-free-paths between points $\overrightarrow{r}'$ and $\overrightarrow{r}$, i.e.,

$$\tau(\overrightarrow{r}', \overrightarrow{r}) = \int_0^1 \exp\left\{ -\sigma_t \left[\overrightarrow{r} + (\overrightarrow{r}' - \overrightarrow{r}) s \right] \right\} \, ds \,. \tag{21b}$$

Finally, we can obtain a kernel for incident surface fluxes by considering only the boundary term in Eq. (17):

$$\phi(\overrightarrow{r}) = \int_{4\pi} \psi(\overrightarrow{r} - s_b \overrightarrow{\Omega}, \overrightarrow{\Omega}) \exp\left[- \int_0^{s_b} \sigma_t(s') \, ds' \right] \, d\Omega \,. \tag{22}$$

Dividing and multiplying the integrand in Eq. (22) by s_b^2, we get

$$\phi(\overrightarrow{r}) = \int_{4\pi} \frac{\psi(\overrightarrow{r} - s_b \overrightarrow{\Omega}, \overrightarrow{\Omega})}{s_b^2} \exp\left[- \int_0^{s_b} \sigma_t(s') \, ds' \right] \, s_b^2 d\Omega \,. \tag{23}$$

It is shown in Appendix B that

$$s_b^2 \, d\Omega = | \overrightarrow{\Omega} \cdot \overrightarrow{n} | \, dA \,, \tag{24}$$

where $\overrightarrow{n}$ is the outward-directed surface normal. Thus we can re-write Eq. (23) in surface kernel form as

$$\phi(\overrightarrow{r}) = \oint_{\Gamma} \frac{\psi(\overrightarrow{r}', \overrightarrow{\Omega}_0) | \overrightarrow{\Omega}_0 \cdot \overrightarrow{n} |}{\| \overrightarrow{r}' - \overrightarrow{r} \|^2} \exp\left[-\tau(\overrightarrow{r}', \overrightarrow{r}) \right] \, dA' \,, \tag{25}$$

where Γ denotes the surface of the transport domain.

Combining Eqs. (21a) and (25), we get the full kernel form of Eq. (17):

$$\phi(\overrightarrow{r}) = \oint_{\Gamma} \frac{\psi(\overrightarrow{r}', \overrightarrow{\Omega}_0)|\overrightarrow{\Omega}_0 \cdot \overrightarrow{n}|}{\|\overrightarrow{r}' - \overrightarrow{r}\|^2} \exp\left[-\tau(\overrightarrow{r}', \overrightarrow{r})\right] \, dA' + \int_{\mathcal{D}} \frac{\mathcal{Q}(\overrightarrow{r}'', \overrightarrow{\Omega}_0)}{\|\overrightarrow{r}'' - \overrightarrow{r}\|^2} \exp\left[-\tau(\overrightarrow{r}'', \overrightarrow{r})\right] \, dV''. \tag{26}$$